

On the steady state probability distribution of nonequilibrium stochastic systems

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A driven stochastic system in a constant temperature heat bath relaxes into a steady state which is characterized by the steady state probability distribution. We investigate the relationship between the driving force and the steady state probability distribution. We adopt the force decomposition method in which the force is decomposed as the sum of a gradient of a steady state potential and the remaining part. The decomposition method allows one to find a set of force fields each of which is compatible to a given steady state. Such a knowledge provides a useful insight on stochastic systems especially in the nonequilibrium situation. We demonstrate the decomposition method in stochastic systems under overdamped and underdamped dynamics and discuss the connection between them.

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I. INTRODUCTION

Systems in contact with a thermal heat bath are described by the probability distribution [1–3]. When the dynamics satisfies the detailed balance (DB), a system evolves into the equilibrium state whose probability distribution is given by the Boltzmann distribution. Recently, much attention has been paid to nonequilibrium systems since many interesting small-sized systems, biological systems, and complex systems are usually driven out of equilibrium. In contrast to equilibrium systems, the DB is broken in nonequilibrium systems. Hence the steady state distribution deviates from the Boltzmann distribution. It is one of the important tasks of nonequilibrium statistical mechanics to characterize the nonequilibrium steady state.

The exact steady state probability distribution is known for only a few cases. Some of stochastic systems governed by the master equation, such as the asymmetric simple exclusion process [4] and the zero range process [5–7], are exactly solvable. For overdamped Langevin equation systems, it is easily found on a one-dimensional ring [1]. In higher dimensions with a linear force, the probability distribution can be written in terms of an anti-symmetric matrix whose elements are given by the solution of a set of algebraic equations [8]. For underdamped Langevin equation systems, several classes of solvable models are found in one spatial dimension [9, 10]. Besides the exceptional solvable cases, it is hard to obtain explicitly the probability distribution of general nonequilibrium systems.

In this paper, we consider stochastic systems which are in thermal contact with a single heat bath and driven by an external force. Given the difficulty of finding the steady state, we focus on the algebraic relationship between the driving force and the steady state probability distribution. Our approach is based on the decomposition of the driving force into two parts. The decomposition has been recognized as a useful tool to characterize the steady state [1, 8, 11, 12]. Once the steady

state is known, the force can be uniquely decomposed into two, each of which reflects the probability current in the steady state.

Applicability of the decomposition method has been limited because it requires the knowledge of the steady state probability distribution in advance. We elaborate more on the decomposition method to reveal some lights it can shed on the steady state of general stochastic systems. Our findings are listed as below: For the overdamped dynamics, it yields a first-order nonlinear differential equation for the steady state distribution in comparison with the second-order linear differential equation obtained from the Fokker-Planck equation. It allows us to find a solvability condition for a linear diffusion system. When the force matrix is *normal*, the steady state probability distribution is found explicitly. For general stochastic systems, overdamped or underdamped, it also provides a systematic way to find a class of driving forces that share the same steady state. Such an information is useful in understanding the extent to which a given steady state covers different physical systems as well as the allowed forms of the steady state probability distribution. This study shows how overdamped dynamics is achieved from underdamped dynamics. We also find interesting solvable nonequilibrium models which can be used to examine the recent developments such as the modified fluctuation dissipation relations [13–15] and the nonequilibrium fluctuation theorems [16–20].

The paper is organized as follows. In Sec. II, we review the Langevin equation and the Fokker-Planck equation formalisms and then introduce the force decomposition method. It is then applied to the overdamped systems in Sec. III and to the underdamped systems in Sec. IV. We conclude the paper with summary in Sec. V.

II. FORCE DECOMPOSITION

In this section, we briefly review the Langevin equation and the Fokker-Planck equation formalisms in order to

set the notation. We refer the reader to Refs. [1–3] for the detailed review. The Langevin equation for a stochastic system with n real variables $\mathbf{q} = (q_1, q_2, \dots, q_n)^T \in \mathbb{R}^n$, represented as a column vector, is written as

$$\dot{q}_i = h_i(\mathbf{q}) + \sum_j g_{ij} \zeta_j(t), \quad (1)$$

where $\boldsymbol{\zeta}(t) = (\zeta_1(t), \dots, \zeta_n(t))^T$ are the Gaussian random noises satisfying

$$\langle \zeta_i(t) \rangle = 0, \quad \langle \zeta_i(t) \zeta_j(t') \rangle = 2\delta_{ij} \delta(t - t'), \quad (2)$$

h_i 's represent the deterministic part of the time evolution, and g_{ij} 's represent the noise strength. The superscript T denotes the transpose. The variable q_i may represent a component of the position or the velocity vector. We are interested in the steady state of a stochastic system in a thermal heat bath, so both h_i and g_{ij} are assumed to be independent of t and g_{ij} does not depend on \mathbf{q} .

Let $P(\mathbf{q}, t)$ denote the probability distribution of the system at time t . It evolves in time following the Fokker-Planck equation [1]

$$\frac{\partial}{\partial t} P(\mathbf{q}, t) = \left[-\sum_{i=1}^n \frac{\partial}{\partial q_i} D_i + \sum_{i,j=1}^n \frac{\partial^2}{\partial q_i \partial q_j} D_{ij} \right] P, \quad (3)$$

where $\mathbf{D} = (D_1, \dots, D_n)^T$ is the drift vector with

$$D_i = h_i(\mathbf{q}) \quad (4)$$

and $\mathbf{D} = \{D_{ij}\}$ is the diffusion matrix with

$$D_{ij} = \sum_k g_{ik} g_{jk}. \quad (5)$$

The Fokker-Planck equation can be casted into the continuity equation

$$\frac{\partial P}{\partial t} + \sum_{i=1}^n \frac{\partial J_i}{\partial q_i} = 0 \quad (6)$$

with the probability current density

$$J_i(\mathbf{q}, t) = \left(D_i - \sum_j \frac{\partial}{\partial q_j} D_{ij} \right) P(\mathbf{q}, t). \quad (7)$$

When t goes infinity, the system reaches the steady state. The steady state probability distribution will be denoted as

$$P_{st}(\mathbf{q}) = e^{-\phi(\mathbf{q})} \quad (8)$$

with the steady state potential $\phi(\mathbf{q})$. Obviously, it is given by the solution of

$$\left[-\sum_{i=1}^n \frac{\partial}{\partial q_i} D_i + \sum_{i,j=1}^n \frac{\partial^2}{\partial q_i \partial q_j} D_{ij} \right] P_{eq}(\mathbf{q}) = 0. \quad (9)$$

The steady-state probability current density is given by

$$J_{st,i}(\mathbf{q}) = e^{-\phi(\mathbf{q})} \left(D_i + \sum_j D_{ij} \frac{\partial \phi}{\partial q_j} \right), \quad (10)$$

where we used that the D_{ij} 's are independent of \mathbf{q} . The steady-state condition of (9) becomes

$$\sum_i \frac{\partial J_{st,i}}{\partial q_i} = 0. \quad (11)$$

The expression (10) suggests that the drift coefficient may be decomposed as $D_i = D_i^{(s)} + D_i^{(a)}$ with [1]

$$D_i^{(s)}(\mathbf{q}) = -\sum_j D_{ij} \frac{\partial \phi}{\partial q_j}, \quad (12)$$

$$D_i^{(a)}(\mathbf{q}) = D_i - D_i^{(s)}. \quad (13)$$

The steady-state current is determined by $\mathbf{D}_i^{(a)}$;

$$J_{st,i}(\mathbf{q}) = D_i^{(a)}(\mathbf{q}) e^{-\phi(\mathbf{q})}. \quad (14)$$

We will call $\mathbf{D}^{(a)} = (D_1^{(a)}, \dots, D_n^{(a)})^T$ and $\mathbf{D}^{(s)} = (D_1^{(s)}, \dots, D_n^{(s)})^T$ the *streaming* vector and the *down-hill* vector, respectively.

As stated in Ref. [1], the decomposition is possible only when the steady state potential ϕ is known, which is hard in general. Despite the difficulty, we find that the decomposition is useful in studying the nature of the steady states. Using (11) and (14), we obtain the relation

$$\sum_i \left(\frac{\partial D_i^{(a)}}{\partial q_i} - D_i^{(a)} \frac{\partial \phi}{\partial q_i} \right) = 0. \quad (15)$$

that constrains the streaming part and the steady state potential. We can approach the steady state problem in a different perspective through the relation. These will be pursued for systems under the overdamped and the underdamped dynamics for a Brownian particle coupled to a single heat bath in the following sections.

III. OVERDAMPED DYNAMICS

A Brownian particle in the d -dimensional space is driven by an external force in a thermal heat bath characterized by the damping coefficient γ and the temperature T . The overdamped dynamics is governed by the Langevin equation

$$\gamma \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \boldsymbol{\xi}(t) \quad (16)$$

where $\mathbf{x} = (x_1, \dots, x_d)^T \in \mathbb{R}^d$ denotes the position vector and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))^T$ denotes an external force. The thermal noise $\boldsymbol{\xi}(t) = (\xi_1(t), \dots, \xi_d(t))^T$ satisfies

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = 2\gamma k_B T \delta_{ij} \delta(t - t'). \quad (17)$$

This system corresponds to (1) with $n = d$, $\mathbf{q} = \mathbf{x}$, $h_i = f_i/\gamma$, and $g_{ij} = \sqrt{k_B T/\gamma} \delta_{ij}$. In terms of the Fokker-Planck equation, the drift coefficient and the diffusion matrix are given by

$$D_i = \frac{f_i}{\gamma} \text{ and } D_{ij} = \frac{k_B T}{\gamma} \delta_{ij}. \quad (18)$$

Hereafter, the Boltzmann constant k_B will be set to unity.

The drift vector is proportional to the driving force. Thus one can decompose the total force $\mathbf{f}(\mathbf{x})$, instead of the drift vector, as

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}_c(\mathbf{x}) + \mathbf{f}_{nc}(\mathbf{x}), \quad (19)$$

where

$$\mathbf{f}_c(\mathbf{x}) = -T \nabla \phi(\mathbf{x}) \quad (20)$$

and

$$\mathbf{f}_{nc}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + T \nabla \phi(\mathbf{x}) \quad (21)$$

with the gradient operator ∇ . Note that $\mathbf{f}_c = \gamma \mathbf{D}^{(s)}$ and $\mathbf{f}_{nc} = \gamma \mathbf{D}^{(a)}$. From (11), we require the steady state condition

$$\nabla \cdot (e^{-\phi} \mathbf{f}_{nc}) = 0. \quad (22)$$

It can be rewritten as

$$(\mathbf{f}_c + T \nabla) \cdot \mathbf{f}_{nc} = 0 \quad (23)$$

or

$$(\mathbf{f} - \mathbf{f}_{nc} + T \nabla) \cdot \mathbf{f}_{nc} = 0. \quad (24)$$

It is a nonlinear first-order partial-differential equation for $\mathbf{f}_{nc}(\mathbf{x})$. Once the solution is found to a given force $\mathbf{f}(\mathbf{x})$, $\mathbf{f}_c(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{f}_{nc}(\mathbf{x}) = -T \nabla \phi(\mathbf{x})$ yields the steady state probability distribution $P_{st}(\mathbf{x}) = e^{-\phi(\mathbf{x})}$.

The decomposition is trivial when the force is conservative in the form of $\mathbf{f} = -\nabla V(\mathbf{x})$ with a scalar function $V(\mathbf{x})$. The choice of $\mathbf{f}_c = \mathbf{f}$ and $\mathbf{f}_{nc} = 0$ satisfies (23). Hence, the steady state distribution is given by the equilibrium Boltzmann distribution with $\phi(\mathbf{x}) = \beta V(\mathbf{x})$ up to a normalization constant with $\beta = 1/T$. When the force is nonconservative, the decomposition becomes nontrivial.

Decomposition of a given vector field $\mathbf{f}(\mathbf{x})$ is an interesting mathematical problem. For instance, the Hodge theory [21, 22] guarantees that a doubly-periodic vector field $\mathbf{f}(x, y)$ on the xy -plane, namely a vector field on a torus, can be written uniquely as the sum of the three vector fields;

$$\mathbf{f}(x, y) = \nabla \psi(x, y) + \mathbf{k} \times \nabla \varphi(x, y) + (c_1, c_2), \quad (25)$$

where ψ and φ are smooth doubly periodic functions on \mathbb{R}^2 , \mathbf{k} is the unit vector in the z direction perpendicular to the xy -plane, \times denotes the cross product, and (c_1, c_2)

is a constant vector field. Note that the second and third types of vector fields are divergence free. Let us consider the case when $\psi(\mathbf{x}) = c\varphi(\mathbf{x})$ for a constant c , and $c_1 = c_2 = 0$. In this case, one can set $\mathbf{f}_c = \nabla \psi$ and $\mathbf{f}_{nc} = \mathbf{k} \times \nabla \varphi$, as they are pointwise perpendicular ($\mathbf{f}_c \cdot \mathbf{f}_{nc} = 0$) and \mathbf{f}_{nc} is divergence free ($\nabla \cdot \mathbf{f}_{nc} = 0$). Such a system has the steady state potential $\phi(x, y) = -\beta \psi(x, y)$.

A. Driven particle in one-dimensional ring

The decomposition condition in (24) can be solved exactly in one dimension. Consider a particle in a one-dimensional ring $0 \leq x \leq L$ which is subject to a periodic potential $V(x) = V(x + L)$ and a uniform driving force f_0 so that $f(x) = -V'(x) + f_0$. The prime ' denotes the derivative with respect to x . The periodic boundary condition is imposed. This problem was studied thoroughly in e.g. Refs. [1, 23].

In the conventional approach [1], the steady state is found from the probability conservation. From (6) and (7), the steady state current

$$J_{st} = [f(x)P_{st}(x) - TP'_{st}(x)]/\gamma \quad (26)$$

should be a constant independent of x in one dimension. Introducing a multi-valued pseudo-potential function

$$\tilde{V}(x) = V(x) - f_0 x \quad (27)$$

and multiplying the both sides of (26) with an integrating factor $e^{\beta \tilde{V}(x)}$, one obtains

$$\left[e^{\beta \tilde{V}(x)} P_{st}(x) \right]' = -\beta \gamma J_{st} e^{\beta \tilde{V}(x)}. \quad (28)$$

It is integrated to yield the solution

$$P_{st}(x) = \beta \gamma J_{st} e^{-\beta \tilde{V}(x)} \left[c - \int_0^x e^{\beta \tilde{V}(y)} dy \right] \quad (29)$$

with an integration constant c which is determined from the periodic boundary condition $P_{st}(0) = P_{st}(L)$:

$$c = \frac{e^{-\beta \tilde{V}(L)} \int_0^L e^{\beta \tilde{V}(y)} dy}{e^{-\beta \tilde{V}(L)} - e^{-\beta \tilde{V}(0)}} = \frac{\int_0^L e^{\beta(V(y) - f_0 y)} dy}{1 - e^{-\beta f_0 L}} \quad (30)$$

The steady state current J_{st} is determined from the normalization $\int P_{st}(x) dx = 1$, which yields

$$J_{st} = \frac{T/\gamma}{\int_0^L dx e^{-\beta(V(x) - f_0 x)} \left(c - \int_0^x dy e^{\beta(V(y) - f_0 y)} \right)}. \quad (31)$$

The same steady state distribution is reproduced from (24), which becomes

$$T f'_{nc} - f_{nc}^2 + f(x) f_{nc} = 0. \quad (32)$$

The transformation $g(x) = 1/f_{nc}(x)$ linearizes the differential equation to the form

$$-T g' + f g = 1. \quad (33)$$

Note that this equation is almost the same as that for $P_{eq}(x)$ in (26). Following the same procedure, we obtain that

$$g(x) = \beta e^{-\beta \tilde{V}(x)} \left[c - \int_0^x e^{\beta \tilde{V}(y)} dy \right] \quad (34)$$

where c is given in (30). The steady state distribution $P_{st}(x) = e^{-\phi(x)}$ is then determined from the relation $f_c(x) = -T\phi'(x) = f(x) - f_{nc}(x) = f(x) - 1/g(x)$. Using the solution for $g(x)$, we find that

$$\phi(x) = \beta \tilde{V}(x) - \ln \left| c - \int_0^x e^{\beta \tilde{V}(y)} dy \right| + \phi_0 \quad (35)$$

with a normalization constant ϕ_0 . This solution is identical to the one in (29). This example shows that the decomposition method works equally well as the conventional method for one-dimensional systems.

B. Linear diffusion systems

Consider a linear diffusion system where the force is linear in $\mathbf{x} \in \mathbb{R}^d$:

$$\mathbf{f}(\mathbf{x}) = \mathbf{F}\mathbf{x}, \quad (36)$$

where \mathbf{F} is a $d \times d$ force matrix whose elements are constant. This system is also called the d -dimensional Ornstein-Uhlenbeck process. It attracts a lot of recent interests for the study of nonequilibrium fluctuations [8, 24, 25]. Our task is to find the steady state distribution function $P_{st}(\mathbf{x}) = e^{-\phi(\mathbf{x})}$ using the decomposition method.

We write the conservative part of (36) as $\mathbf{f}_c = \mathbf{F}_c\mathbf{x}$ with a symmetric matrix $\mathbf{F}_c = \mathbf{F}_c^T$ and the nonconservative part as $\mathbf{f}_{nc} = \mathbf{F}_{nc}\mathbf{x}$ with $\mathbf{F}_{nc} = \mathbf{F} - \mathbf{F}_c$. Then, the steady state condition (23) becomes

$$\mathbf{x}^T \mathbf{F}_c \mathbf{F}_{nc} \mathbf{x} = -T \text{Tr} \mathbf{F}_{nc}. \quad (37)$$

This is valid at all \mathbf{x} only when \mathbf{F}_{nc} is traceless and $\mathbf{A} \equiv \mathbf{F}_c \mathbf{F}_{nc}$ is anti-symmetric:

$$\text{Tr} \mathbf{F}_{nc} = 0 \text{ and } \mathbf{F}_c \mathbf{F}_{nc} = -\mathbf{F}_{nc}^T \mathbf{F}_c. \quad (38)$$

This condition implies that the nonconservative part \mathbf{f}_{nc} is divergence free ($\nabla \cdot \mathbf{f}_{nc} = \text{Tr} \mathbf{F}_{nc} = 0$) and perpendicular to the conservative part ($\mathbf{f}_c \cdot \mathbf{f}_{nc} = 0$). Once the decomposition is found, the steady state distribution function is given by $P_{st}(\mathbf{x}) = e^{-\phi(\mathbf{x})}$ with

$$\phi(\mathbf{x}) = -\frac{1}{2T} \mathbf{x}^T \mathbf{F}_c \mathbf{x} \quad (39)$$

up to a normalization constant.

The decomposition condition may be found by solving the set of algebraic equations in (38) for the elements of \mathbf{F}_c and $\mathbf{F}_{nc} = \mathbf{F} - \mathbf{F}_c$. Kwon *et al* [8] considered general

linear diffusion systems with nondiagonal diffusion matrix. Their formalism yields that the conservative part is written in the form of $\mathbf{F}_c = (1 + \mathbf{Q}/T)^{-1} \mathbf{F}$ with an anti-symmetric matrix \mathbf{Q} [8]. The anti-symmetric matrices \mathbf{Q} and $\mathbf{A} = \mathbf{F}_c \mathbf{F}_{nc}$ are related as $\mathbf{Q} = T \mathbf{F}_c^{-1} \mathbf{A} \mathbf{F}_c^{-1}$.

The decomposition is trivial for equilibrium systems where the force is conservative with a symmetric \mathbf{F} ($\mathbf{F}_c = \mathbf{F}$ and $\mathbf{F}_{nc} = 0$). Besides the equilibrium case, the decomposition method allows one to find another solvable class for the force matrix. Suppose that the force matrix \mathbf{F} is *normal* [26], which means that

$$\mathbf{F} \mathbf{F}^T = \mathbf{F}^T \mathbf{F}. \quad (40)$$

A matrix is then decomposed as

$$\mathbf{F}_c = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) \text{ and } \mathbf{F}_{nc} = \frac{1}{2}(\mathbf{F} - \mathbf{F}^T). \quad (41)$$

Note that \mathbf{F}_c is symmetric and \mathbf{F}_{nc} is traceless. Furthermore, the normal condition in (40) guarantees that

$$\mathbf{F}_c \mathbf{F}_{nc} = \frac{1}{4}(\mathbf{F} + \mathbf{F}^T)(\mathbf{F} - \mathbf{F}^T) = \frac{1}{4}(\mathbf{F}^2 - (\mathbf{F}^T)^2) \quad (42)$$

should be anti-symmetric. Consequently, (41) is the proper decomposition leading to

$$\phi(\mathbf{x}) = -\frac{1}{2T} \mathbf{x}^T \frac{(\mathbf{F} + \mathbf{F}^T)}{2} \mathbf{x} \quad (43)$$

up to a normalization constant.

The normal matrix includes orthogonal ($\mathbf{F} \mathbf{F}^T = \mathbf{I}$), symmetric ($\mathbf{F} = \mathbf{F}^T$), anti-symmetric ($\mathbf{F} = -\mathbf{F}^T$) matrices, and others. For instance,

$$\mathbf{F} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}. \quad (44)$$

is an example of the normal matrix. The corresponding force $\mathbf{f} = \mathbf{F}\mathbf{x}$ attracts the Brownian particle toward the origin with an additional rotational driving. Due to the normality, the steady state potential is given by $\phi(\mathbf{x}) = -\mathbf{x}^T(\mathbf{F} + \mathbf{F}^T)\mathbf{x}/(4T) = (x^2 + y^2 + z^2 + xy + yz + zx)/(2T)$.

C. Force fields family

We have considered the special cases where the decomposition can lead to the steady state explicitly, which is not possible for general cases. In this subsection, we propose a different perspective in which the decomposition plays an interesting role. Instead of solving for the steady state potential to a given force field $\mathbf{f}(\mathbf{x})$, we try to find a force field that leads to a given steady state $P_{st}(\mathbf{x}) = e^{-\phi(\mathbf{x})}$. The steady state probability distribution can be measured experimentally [27–29]. It would be interesting if one could reconstruct a force field from a measurement.

To a given steady state distribution $P_{st}(\mathbf{x}) = e^{-\phi(\mathbf{x})}$, the nonconservative part $\mathbf{f}_{nc}(\mathbf{x})$ should satisfy the steady state condition (22). The solution is not unique. Let $\mathbf{B}(\mathbf{x})$ be any divergence-free vector field. Then, (22) suggests that the nonconservative force should be of the form

$$\mathbf{f}_{nc}(\mathbf{x}) = \lambda e^{\phi(\mathbf{x})} \mathbf{B}(\mathbf{x}) \quad (45)$$

with an arbitrary constant λ . Hence, any system with a force field

$$\mathbf{f}(\mathbf{x}) = -T\nabla\phi(\mathbf{x}) + \lambda e^{\phi(\mathbf{x})} \mathbf{B}(\mathbf{x}) \quad (46)$$

shares the same steady state distribution.

There exist infinitely-many (depending on the choice of $\mathbf{B}(\mathbf{x})$) one-parameter (represented by λ) families of the force fields to a given steady state potential $\phi(\mathbf{x})$. The parameter λ represents the strength of the nonequilibrium driving. In Ref. [30], it was shown that the systems with λ and $-\lambda$ are dual to each other with respect to time reversal (see also Ref. [31]).

In order to gain an intuitive understanding, consider a two-dimensional system having a steady state potential

$$\phi(x, y) = \beta \left[\frac{1}{4}(x^2 - 1)^2 + \frac{y^2}{2} \right] + \phi_0 \quad (47)$$

with a normalization constant ϕ_0 . It is most probable to find the particle at $(x^*, y^*) = (\pm 1, 0)$. Such a steady state is realized in an equilibrium system driven by a conservative force $\mathbf{f}(\mathbf{x}) = -T\nabla\phi = (-x^3 + x, -y)$. The most probable positions coincide with the stable fixed point of the conservative force. As a divergence-free field, we choose

$$\mathbf{B}(\mathbf{x}) = e^{-\frac{r^4}{4}} \begin{pmatrix} -y \\ x \end{pmatrix} \quad (48)$$

with $r = \sqrt{x^2 + y^2}$. Then, any total force of the form

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x^3 + x \\ -y \end{pmatrix} + \lambda e^{-\frac{(x^2-1)^2 + y^2 + 1}{4}} \begin{pmatrix} -y \\ x \end{pmatrix} \quad (49)$$

has the same steady state potential in (47). The force field lines are drawn in Fig. 1 at a few values of λ . One observes a quantitative and qualitative changes as λ varies. The stable fixed points move with λ deviating from the most probable points at (x^*, y^*) . The fixed points even undergo a bifurcation at $|\lambda| = 1$, which leaves behind a single stable fixed point at $(x, y) = (0, 0)$ for $|\lambda| > 1$ (see Fig. 2). It is remarkable that those force fields with different fixed point structures share the same steady state. The discrepancy between the most probable point and the stable fixed point was reported with a perturbative calculation in Ref. [12]. The decomposition method confirms that the discrepancy is a general property of nonequilibrium systems driven by a nonlinear force.

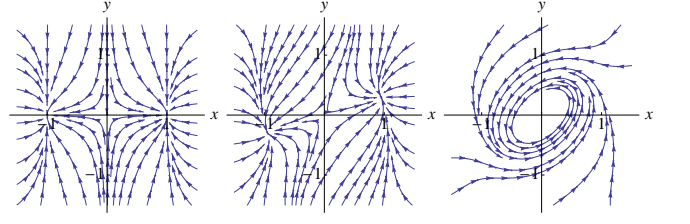


FIG. 1. Field lines of the force in (49) with $\lambda = 0.0$ (left), 0.5 (center), and 2.0 (right).

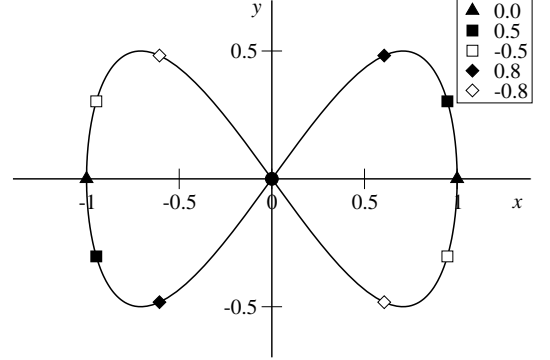


FIG. 2. Trajectory of the stable fixed points. The fixed points at several values of λ are marked with symbols. The origin $(0, 0)$ marked by the closed circle is the unique stable fixed point when $|\lambda| \geq 1$.

IV. UNDERDAMPED DYNAMICS

The underdamped dynamics of a Brownian particle of mass m is governed by the Langevin equation

$$\dot{\mathbf{x}} = \mathbf{v} \quad (50)$$

$$m\dot{\mathbf{v}} = -\gamma\mathbf{v} + \mathbf{f}(\mathbf{x}, \mathbf{v}) + \boldsymbol{\xi} \quad (51)$$

for the position $\mathbf{x} = (x_1, \dots, x_d)^T \in \mathbb{R}^d$ and the velocity $\mathbf{v} = (v_1, \dots, v_d)^T \in \mathbb{R}^d$. The force \mathbf{f} may depend on both \mathbf{x} and \mathbf{v} in general. For convenience, we introduce a notation $\mathbf{q} = (q_1, \dots, q_{2d})^T = (\mathbf{x}, \mathbf{v})^T$ where $q_i = x_i$ and $q_{d+i} = v_i$ for $1 \leq i \leq d$. Then, the Langevin equations take the form of (1). The corresponding Fokker-Planck equation has the drift coefficients

$$D_i = D_{x_i} = v_i \text{ and } D_{d+i} = D_{v_i} = -\frac{\gamma}{m}v_i + \frac{f_i}{m} \quad (52)$$

with $1 \leq i \leq d$. It is represented as a $(2d)$ -dimensional column vector

$$\mathbf{D} = \begin{pmatrix} D_{\mathbf{x}} \\ D_{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ -\frac{\gamma}{m}\mathbf{v} + \frac{1}{m}\mathbf{f} \end{pmatrix}. \quad (53)$$

The diffusion matrix $\mathbf{D} = \{D_{ij}\}$ has the elements $D_{ij} = D_{i,d+j} = D_{d+i,j} = 0$ and

$$D_{d+i,d+j} = D_{v_i,v_j} = \frac{\gamma T}{m^2} \delta_{ij} \quad (54)$$

for $1 \leq i, j \leq d$. It is represented as a $(2d) \times (2d)$ matrix

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\gamma T}{m^2} \mathbf{1} \end{pmatrix}, \quad (55)$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the $d \times d$ null and identity matrices, respectively.

We want to find a force field $\mathbf{f}(\mathbf{x}, \mathbf{v})$ that leads to a given steady state $P_{st}(\mathbf{x}, \mathbf{v}) = e^{-\phi(\mathbf{x}, \mathbf{v})}$. According to (12) and (13), we decompose the drift coefficient as the sum of

$$\begin{aligned} \mathbf{D}^{(s)} &= -\mathbf{D} \nabla \phi = \begin{pmatrix} \mathbf{0} \\ -\frac{\gamma T}{m^2} \nabla_{\mathbf{v}} \phi \end{pmatrix} \\ \mathbf{D}^{(a)} &= \begin{pmatrix} \mathbf{D}_{\mathbf{x}}^{(a)} \\ \mathbf{D}_{\mathbf{v}}^{(a)} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ -\frac{\gamma}{m} \mathbf{v} + \frac{1}{m} \mathbf{f} + \frac{\gamma T}{m^2} \nabla_{\mathbf{v}} \phi \end{pmatrix}, \end{aligned} \quad (56)$$

where ∇ is the gradient operation in the (\mathbf{x}, \mathbf{v}) space while $\nabla_{\mathbf{v}}$ is the gradient operator acting on the subspace of \mathbf{v} . The steady state condition requires that

$$\nabla \cdot (\mathbf{D}^{(a)} e^{-\phi}) = \nabla_{\mathbf{x}} \cdot (\mathbf{D}_{\mathbf{x}}^{(a)} e^{-\phi}) + \nabla_{\mathbf{v}} \cdot (\mathbf{D}_{\mathbf{v}}^{(a)} e^{-\phi}) = 0,$$

where $\nabla_{\mathbf{x}}$ is the gradient operator acting on the subspace of \mathbf{x} .

The general solution for $\mathbf{D}^{(a)}$ is written as

$$\mathbf{D}^{(a)} = e^{\phi} \mathbf{B} \quad (58)$$

where $\mathbf{B}(\mathbf{x}, \mathbf{v}) \equiv \begin{pmatrix} \mathbf{B}_{\mathbf{x}} \\ \mathbf{B}_{\mathbf{v}} \end{pmatrix}$ is a divergence-free ($\nabla_{\mathbf{x}} \cdot \mathbf{B}_{\mathbf{x}} + \nabla_{\mathbf{v}} \cdot \mathbf{B}_{\mathbf{v}} = 0$) field.

In contrast to the overdamped case, the auxiliary field \mathbf{B} is subject to the additional kinetic constraint that $\mathbf{D}_{\mathbf{x}}^{(a)} = \mathbf{v}$, which sets $\mathbf{B}_{\mathbf{x}} = e^{-\phi} \mathbf{v}$. Hence, the divergence-free condition becomes

$$\nabla_{\mathbf{v}} \cdot \mathbf{B}_{\mathbf{v}} = -\nabla_{\mathbf{x}} \cdot \mathbf{B}_{\mathbf{x}} = \mathbf{v} \cdot (\nabla_{\mathbf{x}} \phi) e^{-\phi}. \quad (59)$$

With $\mathbf{B}_{\mathbf{v}}$ satisfying (59), the force field is given by

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = \gamma \mathbf{v} - \frac{\gamma T}{m} \nabla_{\mathbf{v}} \phi + m \mathbf{B}_{\mathbf{v}} e^{\phi}. \quad (60)$$

In terms of $\mathbf{D}_{\mathbf{v}}^{(a)} = \mathbf{B}_{\mathbf{v}} e^{\phi}$, (59) and (60) become

$$\nabla_{\mathbf{v}} \cdot \mathbf{D}_{\mathbf{v}}^{(a)} - \mathbf{D}_{\mathbf{v}}^{(a)} \cdot (\nabla_{\mathbf{v}} \phi) = \mathbf{v} \cdot (\nabla_{\mathbf{x}} \phi) \quad (61)$$

and

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = \gamma \mathbf{v} - \frac{\gamma T}{m} \nabla_{\mathbf{v}} \phi + m \mathbf{D}_{\mathbf{v}}^{(a)}. \quad (62)$$

We compare the overdamped and the underdamped cases. In the former case, the total force is decomposed into the conservative and nonconservative parts. The nonconservative part \mathbf{f}_{nc} is determined up to an arbitrary parameter λ (see (46)). Thus, any nonequilibrium system characterized by a finite λ finds the corresponding equilibrium system ($\lambda = 0$) that shares the same steady state. In the latter case, the decomposition does

not separate the force into the sum of conservative and nonconservative parts. Since (61) is an inhomogeneous equation, $m \mathbf{D}_{\mathbf{v}}^{(a)}$ is given by the sum of the homogeneous solution up to a multiplicative factor and the specific particular solution. Hence, the correspondence between the steady state and the driving force is more restrictive in the underdamped dynamics due to the particular solution. The correspondence is investigated further in the following subsections.

A. Boltzmann distribution

When the Brownian particle is driven by a velocity-independent conservative force $\mathbf{f}(\mathbf{x}) = -\nabla_{\mathbf{x}} V(\mathbf{x})$, the system reaches the equilibrium Boltzmann distribution

$$\phi(\mathbf{x}, \mathbf{v}) = \beta \left[\frac{m}{2} \mathbf{v}^2 + V(\mathbf{x}) \right]. \quad (63)$$

It is interesting to see whether the Boltzmann distribution (63) can be also realized in other forces than the conservative forces.

For the Boltzmann distribution, $\nabla_{\mathbf{v}} \phi = \beta m \mathbf{v}$ and $\mathbf{D}_{\mathbf{v}}^{(a)} = \mathbf{f}/m$ from (62). Hence, the force should satisfy

$$\nabla_{\mathbf{v}} \cdot \mathbf{f} - \beta m \mathbf{v} \cdot \mathbf{f} = m \beta \mathbf{v} \cdot (\nabla_{\mathbf{x}} V(\mathbf{x})). \quad (64)$$

The general solution is given by

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = \lambda e^{\beta \frac{m \mathbf{v}^2}{2}} \mathbf{C}(\mathbf{x}, \mathbf{v}) - \nabla_{\mathbf{x}} V(\mathbf{x}), \quad (65)$$

where $\mathbf{C}(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^d$ is a vector field satisfying $\nabla_{\mathbf{v}} \cdot \mathbf{C}(\mathbf{x}, \mathbf{v}) = 0$ and λ is an arbitrary parameter. The first term is the homogeneous solution and the second term is the particular solution of (64).

One lesson from (65) is that there exist infinitely many nonequilibrium forces sharing the same Boltzmann distribution in the steady state. We also find that the Boltzmann distribution requires a velocity dependent force except for the equilibrium case with $\lambda = 0$. If one perturbs an equilibrium system with any velocity-independent nonconservative force, the steady state must deviate from the Boltzmann distribution. This is in sharp contrast to the overdamped system.

A special type of the homogeneous solution of (64) is found by requiring $\nabla_{\mathbf{v}} \cdot \mathbf{f} = \beta m \mathbf{v} \cdot \mathbf{f} = 0$. It yields

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = \lambda \mathbf{H}(\mathbf{x}) \mathbf{v} - \nabla_{\mathbf{x}} V(\mathbf{x}), \quad (66)$$

where $\mathbf{H}(\mathbf{x})$ is an arbitrary antisymmetric matrix. This force is linear in \mathbf{v} and perpendicular to \mathbf{v} . In $d = 3$ dimensions, it is the familiar magnetic force on a charged particle in an inhomogeneous magnetic field.

B. Shifted Boltzmann distribution in one-dimensional ring

As a solvable extension, we consider a shifted Boltzmann distribution

$$\phi(x, v) = \beta \left[\frac{m}{2} (v - \omega(x))^2 + V(x) \right] \quad (67)$$

in a one-dimensional ring of circumference L . The functions $\omega(x+L) = \omega(x)$ and $V(x+L) = V(x)$ are periodic. This form might be a natural extension of steady state potential from the overdamped dynamics to the underdamped dynamics. In order to find a corresponding force field $f(x, v)$, one needs to solve (see (59))

$$\frac{\partial B_v}{\partial v} = \beta v [-m(v - \omega)\omega' + V'] e^{-\frac{\beta m}{2}(v - \omega)^2 - \beta V}. \quad (68)$$

Integrating the equation with respect to v , one obtains

$$B_v = C(x)e^{-\beta V} + \left(\omega'v - \frac{V'}{m}\right)e^{-\phi} + \sqrt{\frac{\pi}{2\beta m}}(\beta\omega V' - \omega')[1 + \text{erf}(Z)]e^{-\beta V} \quad (69)$$

where $Z \equiv \sqrt{\frac{\beta m}{2}}(v - \omega(x))$, $\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ is the error function, and $C(x)$ is an arbitrary function of x introduced as an integration constant. Hence, (60) yields

$$f(x, v) = m e^{Z^2} \left[C(x) + \sqrt{\frac{\pi}{2\beta m}}(\beta\omega V' - \omega') \{1 + \text{erf}(Z)\} \right] + (\gamma\omega + mv\omega' - V'). \quad (70)$$

The factor e^{Z^2} would drive the velocity to infinity. Such an instability is avoided by taking $C(x) = 0$ and

$$\omega(x)V'(x) = Tw'(x), \quad (71)$$

which leads to $V(x) = T \ln |\omega(x)| + V_0$ with an arbitrary constant V_0 . Therefore we conclude that the Brownian particle driven by the force

$$f(x, v) = \gamma\omega(x) - V'(x) + mvw'(x) \quad (72)$$

with the constraint (71) has the steady state potential given in (67).

When $\omega(x) = 0$, the system corresponds to a free Brownian particle. When $\omega(x) = \omega_0$ is a constant, $f(x, v) = \gamma\omega_0$ and $\phi(x, v) = \beta m(v - \omega_0)^2/2$. This system corresponds to a Brownian particle driven by a spatially uniform driving force on a ring. With position-dependent $\omega(x)$, the solvable force should be velocity dependent. The solvability requires that the v -dependent part and the v -independent part in (72) are interwoven intimately. The solvable model could be useful in testing various theoretical concepts such as the fluctuation theorems for nonequilibrium systems, especially with velocity-dependent forces [19, 20].

C. Overdamped limit

The shifted Boltzmann distribution gives a hint how an underdamped system is related to an overdamped system. Overdamped dynamics is achieved by taking $m \rightarrow 0$ limit [32, 33]. In this limit, the force (72) in the previous subsection becomes $f_{od}(x) = \gamma\omega(x) - V'(x)$, which

is independent of v . Comparing (71) with (23), we find that $f_{nc}(x) = \gamma\omega(x)$ and $f_c(x) = -V'(x)$ is the proper decomposition of the force f_{od} in the overdamped limit. Hence, the overdamped system with the force $f_{od}(x)$ has the steady state potential $\phi_{od} = \beta V$. Interestingly, it is the same as the steady state potential in (67) after being averaged over v .

We can generalize the shifted Boltzmann distribution in arbitrary d dimensions. Consider a steady state potential of the form

$$\phi(\mathbf{x}, \mathbf{v}) = \beta \left[\frac{m}{2}(\mathbf{v} - \boldsymbol{\omega}(\mathbf{x}))^2 + V(\mathbf{x}) \right] \quad (73)$$

with a vector field $\boldsymbol{\omega}(\mathbf{x}) = (\omega_1, \dots, \omega_d)^T \in \mathbb{R}^d$ and a scalar field $V(\mathbf{x})$. Repeating the similar algebra as in the previous subsection, we find that the auxiliary vector field $\mathbf{B}_v(\mathbf{x}, \mathbf{v}) = (B_{v_1}, \dots, B_{v_d})$ is given by

$$B_{v_i} = C_i + \left[(\mathbf{v} \cdot \nabla_{\mathbf{x}})\omega_i - \frac{1}{m} \frac{\partial V}{\partial x_i} \right] e^{-\phi} + \sqrt{\frac{\pi}{2\beta m}} (1 + \text{erf} Z_i) \Upsilon e^{-\phi + Z_i^2}, \quad (74)$$

where

$$\Upsilon \equiv (\beta\boldsymbol{\omega} \cdot \nabla_{\mathbf{x}} V - \nabla_{\mathbf{x}} \cdot \boldsymbol{\omega}), \quad (75)$$

$Z_i = \sqrt{\frac{\beta m}{2}}(v_i - \omega_i)$, and $\mathbf{C}(\mathbf{x}, \mathbf{v}) = (C_1, \dots, C_d)^T$ is any vector field satisfying $\nabla_{\mathbf{v}} \cdot \mathbf{C} = 0$. It is straightforward to check this is the solution of (59). Thus, from (60), the corresponding force is given by

$$f_i = [\gamma\omega(\mathbf{x}) - \nabla_{\mathbf{x}} V + m(\mathbf{v} \cdot \nabla_{\mathbf{x}})\boldsymbol{\omega}]_i + C_i e^{\phi} + \sqrt{\frac{\pi m}{2\beta}} (1 + \text{erf} Z_i) \Upsilon e^{Z_i^2}. \quad (76)$$

In order to avoid instability, we will set $C_i = 0$ and $\Upsilon = 0$. Consequently, the force is given by

$$\mathbf{f} = \gamma\boldsymbol{\omega}(\mathbf{x}) - \nabla_{\mathbf{x}} V(\mathbf{x}) + m(\mathbf{v} \cdot \nabla_{\mathbf{x}})\boldsymbol{\omega} \quad (77)$$

with the constraint that $\Upsilon = 0$.

Let us consider the overdamped limit ($m \rightarrow 0$). Then, the force becomes $\mathbf{f}_{od} = \gamma\boldsymbol{\omega} - \nabla_{\mathbf{x}} V$ which is independent of \mathbf{v} . The stability condition $\Upsilon = 0$ guarantees that $\mathbf{f}_c = -\nabla_{\mathbf{x}} V$ and $\mathbf{f}_{nc} = \gamma\boldsymbol{\omega}$ is the proper decomposition of \mathbf{f}_{od} satisfying (23). Hence, the steady state potential of the overdamped system is given by $\phi_{od}(\mathbf{x}) = \beta V(\mathbf{x})$. This is indeed the same steady state potential obtained from (73). It suggests that the shifted Boltzmann distribution is a good approximation of the steady state potential of an underdamped system in the small m limit. The difference between the overdamped dynamics and the underdamped dynamics lies in the \mathbf{v} -dependent term $(\mathbf{v} \cdot \nabla_{\mathbf{x}})\boldsymbol{\omega}$. The origin of this force and its implications are left for a future study.

V. SUMMARY

We have investigated implications of the decomposition method on the relationship between the driving force and the steady state potential of a Brownian particle in a thermal heat bath. The force, or the drift coefficient, can be decomposed as the sum of the down-hill part and the streaming part satisfying the condition (15). The decomposition method reveals some aspects of nonequilibrium steady states. In the overdamped dynamics, any steady state is infinitely degenerate in the sense that it is shared by the family of force fields of the form (46). The most probable points do not coincide with the stable fixed points. In the underdamped dynamics, the correspondence between the force and the steady state potential is restrictive. The Boltzmann-type distribution is real-

ized only in either an equilibrium system or a nonequilibrium system driven by a velocity-dependent force such as a magnetic force. The shifted Boltzmann distribution uniquely determines the corresponding force field under the stability requirement. The shifted Boltzmann distribution is a connection between the overdamped dynamics and the underdamped dynamics. As a byproduct, the decomposition method provides various examples of solvable nonequilibrium systems. Hopefully, they may be useful for the study of nonequilibrium statistical mechanics.

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